Eq. (1). Hence, the piston velocity at $\tilde{x} = \tilde{x}_s$ was found to be

$$\left.\frac{d\tilde{x}}{dt}\right|_s = -\left[\frac{2}{\gamma - 1}\left(1 - \tilde{x}_s^{-(\gamma - 1)}\right)\right]^{1/2} \tag{5}$$

and corresponding gas density

$$\tilde{\rho}_s = 1/\tilde{x}_s \tag{6}$$

In Fig. 2a are shown numerical results for the dimensionless distance $\tilde{x}$ as a function of dimensionless time $\tilde{t}$ for discrete values of $\epsilon$ in the range $0 \leq \epsilon \leq 0.80$ for a gas with $\gamma = \frac{5}{3}$. For $\epsilon = 0$, we have the case of a closed orifice and the bounce takes place exactly at $\tilde{x} = 1$. For increasing values of $\epsilon$, the bounce will occur closer to the insert and at later times. With $\epsilon \sim 0.55$ (for $\tilde{x}_s = 5$), smooth docking is obtained. Piston impact will happen for larger values of $\epsilon$, as shown for $\theta = 0.60$ and $\epsilon = 0.80$. Corresponding pressure-time characteristics are drawn in Fig. 2b. The peak gas pressure in the bounce is found to take smaller values with increasing values of $\epsilon$. Interestingly, the actual reversal of the piston velocity (the bounce) takes place after the pressure has reached its maximum value. In Fig. 2c, the nondimensional measure $\tilde{x}\tilde{\rho}$ of the amount of trapped gas is given as a function of time. Naturally, for $\epsilon = 0$, we have $\tilde{x}\tilde{\rho} = 1$. Values for $\tilde{x}\tilde{\rho}$ decrease continuously with time for all values $\epsilon > 0$. The case of smooth docking for $\epsilon \sim 0.55$ shows a negligible amount of residual gas for $\tilde{t} > 7$.

### Experiment and Concluding Remarks

In experiments conducted with a bypass-piston shock tube,[4] the sonic-orifice insert was placed in the helium compression tube at a location near the turning point of the piston in a calculated bounce. In Fig. 3 is shown a photograph of the employed high-pressure section, the insert (65-mm o.d.) and a piston. Typical bounce parameters were for helium: $x_0 \sim 40$ mm, $p_0 \sim 600$ atm, and $a_0 \sim 5500$ $m/s$ (temperature $T_0 \sim 9000°$K). The pistons used had masses in the range $0.4 < M < 2.0$ kg, and therefore $0.3 \leq \tau_0 \geq 0.6$ ms. The effective area ratio was $A/A^* \sim 35$, giving $\tau_1 \sim 0.5$ ms, and hence $0.6 \leq \epsilon \leq 1.2$. These values for $\epsilon$ were larger than the calculated critical $\epsilon \sim 0.55$ for smooth docking with $\tilde{x}_s = 5$. However, the heavy pistons (with an associated larger value for $\epsilon$) were equipped with conical forebodies with a base diameter equal to the diameter of the orifice in the insert, as is visible in Fig. 3. By gradually plugging the orifice, these cones reduced the effective value of $\epsilon$ when the piston was in the immediate vicinity of the insert and thus diminished the possibility of impact.

The insert was frequently used under conditions which would normally have resulted in destructive piston impacts into the smaller-diameter high-pressure section. With the insert, no such impacts were observed, nor did the steel pistons swell, jam, or seize to the compression tube. The function of the insert was considered most satisfactory and vitally helpful in the successful attempts of achieving shock velocities above 12 km/s in the shock tube.[4]

As theoretically predicted and experimentally verified, the sonic-orifice insert in the free-piston shock-tube driver could completely avoid piston-impact hazards generally connected with a premature rupture of the outlet diaphragm or incorrect gas loading in a free-piston compressor. Use of the insert improves theoretical over-all performance due to an associated increase in entropy of the gas. For a properly designed orifice, a preferred value for the dimensionless parameter $\epsilon$ should be somewhat larger than or equal to the calculated value for smooth docking.

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# Maximum-Minimum Sufficiency and Lagrange Multipliers

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## I. Introduction

Lagrange multipliers are commonly employed when dealing with problems of maxima and minima subject to equality constraints. Their use is usually justified in accordance with some multiplier rule‡ generally phrased in such a way that it provides a convenient constructive formalism for solving problems.

The proof of a multiplier rule found in texts, in particular advanced engineering texts such as Hildebrand (Ref. 1, p. 354), Solkolnikoff, and Redheffer (Ref. 2, p. 256), etc., is often very limited and quite intuitive. Often the general case is not proven and almost universally a discussion of sufficiency conditions for constrained extrema in terms of Lagrange multipliers is omitted.

Because optimizing conditions can be stated very simply and concisely in terms of Lagrange multipliers they remain in popular use. However, perhaps partly because of this extensive use, and partly because of the intuitive notions frequently offered, certain properties have at times been ascribed to the Lagrange multipliers which they do not possess. If we wish to extremize a function $f(x_i)$, $i = 1, \ldots, n$, subject to the constraints $g_j(x_i)$, $j = 1, \ldots, m < n$, a common misconception is the thought that this is identical to extremizing an augmented function $G(x_i,\lambda_j)$ formed by adjoining the constraints of $f$ with the multipliers $\lambda_j$, i.e., $G = f + \lambda_j g_j$. For example, Edelbaum (Ref. 3, p. 11) in his discussion of theory of maxima and minima states, "The use of the augmented function allows a problem with subsidiary conditions to be replaced by a problem without subsidiary conditions. This new problem is amenable to all of the techniques used for solving problems without subsidiary conditions, including sufficiency conditions." Edelbaum is by no means the only author who alludes to this concept; but perhaps he states it most clearly.

Edelbaum's statement in essence represents a multiplier rule. To illustrate the results obtained by using this rule, we first need the following theorem for unconstrained extrema. For proof see Hestenes (Ref. 4, p. 18).

***Theorem 1***

The necessary and sufficient conditions for the function $f(x_i)$ of class $C^2$ on $S$ to take on a local, proper, nonsingular minimum at the point $x_i^p$ on the interior of $S$ is that

$$\partial f/\partial x_i|_{x_i^p} = 0 \tag{1}$$

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‡ A very loose usage is meant here. In other words, a multiplier rule is any rule which postulates optimizing conditions in terms of Lagrange multipliers. Associate AIAA.

and that the matrix

$$[\partial^2 f/\partial x_i \partial x_l]_{x_i^p}, \quad i,l = 1, \ldots, n \tag{2}$$

be positive definite.

According to Edelbaum's rule, and Theorem 1, the necessary and sufficient conditions for $G(x_i,\lambda_j)$ to take on an ordinary (local, proper, nonsingular, interior) minimum are that $\partial G/\partial x_i = \partial G/\partial \lambda_j = 0$ and that the matrix

$$\begin{bmatrix} \dfrac{\partial^2 G}{\partial x_i \partial x_l} & \dfrac{\partial^2 G}{\partial x_i \partial \lambda_j} \\ \dfrac{\partial^2 G}{\partial \lambda_j \partial x_i} & \dfrac{\partial^2 G}{\partial \lambda_j \partial \lambda_k} \end{bmatrix} \equiv \begin{bmatrix} \dfrac{\partial^2 G}{\partial x_i \partial x_l} & \begin{bmatrix} \dfrac{\partial g_j}{\partial x_i} \end{bmatrix}^T \\ \dfrac{\partial g_j}{\partial x_i} & 0 \end{bmatrix} \tag{3}$$

where $i, l = 1, \ldots, n$ and $j, k = 1, \ldots, m$, be positive definite. Unfortunately the latter condition on the matrix (3) is an incorrect result for the extremization of $f$ subject to $g_j = 0$. [See, for example, Hancock (Ref. 5, page 116) for the correctly state result.]

The conditions $\partial G/\partial x_i = \partial G/\partial \lambda_i = 0$ are correct necessary conditions for $f$ to take on an ordinary extreme value with respect to the constraints $g_j$ (indeed, this is as far as most multiplier rules go) and hence, it is evident that normal stationary points of $f$ subject to $g_j$ are also stationary points of $G(x_i,\lambda_j)$. However, there is no guarantee that ordinary extremal points of $f$ subject to $g_i$ are also ordinary extremal points of $G(x_i,\lambda_j)$ and in general they are not. As will be shown, the necessary and sufficient conditions for a local extrema of the function $G(x_i,\lambda_j)$ subject to no constraints are more stringent than for the original problem and hence would often not be satisfied for many problems which possess a perfectly good solution.

In this paper, both the necessary and sufficient conditions are developed without the use of Lagrange multipliers, and only at the last step are certain groupings of terms identified for convenience and for comparison, as multipliers.

## II. Necessary and Sufficient Conditions for Constrained Extrema

The method presented here to develop the necessary and sufficient conditions for constrained extrema will use as a fundamental premise the results of ordinary maxima and minima theory for a function of a single variable. These results are obtained immediately from Theorem 1 and will be stated here for convenience in the form of a lemma.

***Lemma 1***

The necessary and sufficient conditions for the function $f = f(u)$ of class $C^2$ defined in some interval $[a,b]$ to take on a local, proper, nonsingular minimum at the point $u = u^p$ within the interval $(a,b)$ are that $\partial f/\partial u|_{u^p} = 0$ and $\partial^2 f/\partial u^2|_{u^p} > 0$.

Using a control type of notation, the constrained optimization problem is stated as follows: extremize the function $f(y_i,u_k)$, $i = 1, \ldots, n$, and $k = 1, \ldots, m$; subject to the constraints

$$g_j(y_i,u_k) = 0, \quad j = 1, \ldots, n \tag{4}$$

Since there are $n$ constraints equations, the variables $u_k$ take on the role of control parameters and the variables $y_i$ take on the role of state parameters and will be referred to as such henceforth.

Obviously for ordinary maximum-minimum problems, subject to constraints, the choice of control parameters may not be unique. Once a choice has been made however, the determinant, $\partial g_j/\partial y_i$, must be assumed nonzero at the extremal point. Also $f$ and $g_j$ will be assumed to be of class $C^2$ with respect to their arguments.

The function $f$ and the constraints $g_j$ are first transformed into functions of a single parameter $\epsilon$. Let the values of the control at which $f$ takes on ordinary extremum be designated by $u_k^0$. Comparisons of the value of $f$ in the neighborhood of an ordinary extremal value $f^0$ can be made by evaluating $f$ with the following values of $u_k$,

$$u_k = u_k^0 + \epsilon w_k \tag{5}$$

where $w_k$ represent $m$ arbitrarily chosen constants and $\epsilon$ represents a parameter which may be chosen freely during the course of investigation. Using this artifice, all the control parameters may be conveniently varied from their optimal values through the use of a single parameter. Since it has been assumed that $|\partial g_j/\partial y_i|$ is nonzero at ordinary extremal points, the implicit function theorem guarantees that Eq. (4) may be considered to be functionally expressible as

$$y_i = \beta_i(u_k^0 + \epsilon w_k) \tag{6}$$

Through Eqs. (5) and (6), $f$ becomes a function of $\epsilon$ only. To apply the first condition of Lemma 1, the derivative of $f$ with respect to $\epsilon$ is first obtained

$$df/d\epsilon = (\partial f/\partial y_i)h_i + (\partial f/\partial u_k)w_k \tag{7}$$

where $h_i = \partial \beta_i/\partial \epsilon$. The quantities $h_i$ are dependent on the arbitrary $w_k$ as can be seen by noting that since $g_j = 0$ for all variations,

$$dg_j/d\epsilon = (\partial g_j/\partial y_i)h_i + (\partial g_j/\partial u_k)w_k = 0 \tag{8}$$

This equation may be solved for the $h_i$, by first defining the inverse to the matrix $\partial g_i/\partial y_i$. Defining $A_{pj}$, $p = 1, \ldots, n$, such that,

$$A_{pj}\frac{\partial g_j}{\partial y_i} = \delta_{pi}, \quad \delta_{pi} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \tag{9}$$

then multiplying Eq. (8) by $A_{pj}$, setting $p = i$ and solving for $h_i$ yields

$$h_i = -A_{ij}(\partial g_j/\partial u_k)w_k \tag{10}$$

Substituting Eq. (10) into (7) results in

$$df/d\epsilon = [(\partial f/\partial u_k) - (\partial f/\partial y_i)A_{ij}(\partial g_j/\partial u_k)]w_k \tag{11}$$

An operational necessary condition for determining the location of ordinary extremal points will be obtained by setting $df/d\epsilon = 0$. But, first, the second derivative will be examined.

To apply the second condition of Lemma 1, the second derivative of $f$ with respect to $\epsilon$ is required. The derivative of Eq. (11) is simplified by noting that

$$(\partial/\partial u_l)A_{ij} = -A_{iq}(\partial/\partial u_l)(\partial g_q/\partial y_p)A_{pj} \tag{12}$$

and

$$(\partial/\partial y_p)A_{ij} = -A_{iq}(\partial/\partial y_p)(\partial g_q/\partial y_p)A_{pj} \tag{13}$$

Using these relations, the second derivative becomes

$$\begin{aligned} \frac{d^2f}{d\epsilon^2} = {} & \left(\frac{\partial^2 f}{\partial u_l \partial u_k} - \frac{\partial f}{\partial y_i}A_{ij}\frac{\partial^2 g_j}{\partial u_l \partial u_k}\right) w_k w_l + \\ & \left(\frac{\partial^2 f}{\partial y_i \partial u_k} - \frac{\partial z}{\partial y_p}A_{pj}\frac{\partial g_j}{\partial y_i \partial u_k}\right) w_k h_i + \\ & \left(\frac{\partial^2 f}{\partial u_l \partial y_i} - \frac{\partial f}{\partial y_p}A_{pj}\frac{\partial g_j}{\partial u_l \partial y_i}\right) h_i w_l + \\ & \left(\frac{\partial^2 f}{\partial y_j \partial y_i} - \frac{\partial f}{\partial y_p}A_{pq}\frac{\partial^2 g_q}{\partial y_j \partial y_i}\right) h_i h_j \end{aligned} \tag{14}$$

It is convenient to introduce new variables (Lagrange multipliers) defined in the following way. Let

$$\lambda_j = -(\partial f/\partial y_i)A_{ij} \tag{15}$$

and

$$G = f + \lambda_j g_j \tag{16}$$

Then Eqs. (11) and (14) may be written more simply as

$$(df/d\epsilon) = (\partial G/\partial u_k)w_k \tag{17}$$

and

$$\frac{d^2f}{d\epsilon^2} = \frac{\partial^2 G}{\partial u_l \partial u_k} w_k w_l + \frac{\partial^2 G}{\partial y_i \partial u_k} w_k h_i + \frac{\partial^2 G}{\partial u_l \partial y_i} h_i w_l + \frac{\partial^2 G}{\partial y_j \partial y_i} h_i h_j \tag{18}$$

We see from Eq. (18) that the condition $d^2f/d\epsilon^2 > 0$, is a condition on a quadratic form which involves both dependent and independent variations. That is, in evaluating $d^2f/d\epsilon^2$, $h_i$, and $w_k$ are related by the linear Eq. (10). Hence, two options are open before applying Lemma 1. Option 1: Eq. (10) may be used to eliminate the $h_i$ in Eq. (18); Option 2: the following theorem may be used.

***Theorem 2***

The maximum and minimum values of the form $x^T\mathbf{A}x/||x||^2$ subject to the conditions $\mathbf{Bx} = \mathbf{0}$, are the zero's of $D(e)$, where

$$D(e) = \begin{vmatrix} \mathbf{A} - e\mathbf{I} & \mathbf{B}^T \\ \mathbf{B} & 0 \end{vmatrix} \tag{19}$$

For proof see Hestenes (Ref. 4, p. 33).

Hence, it follows if the roots of $D(e) = 0$ are all positive, the quadratic form $x^T\mathbf{A}x$ is positive definite. Under option 1, the necessary and sufficient conditions for constrained extrema reduce to those given in the recent work of Bryson and Ho (Ref. 6, p. 9) and under option 2 the results of Hancock (Ref. 5, p. 116) are obtained.

By making a few observations, a multiplier rule (akin to those normally stated) may now be given which alludes to both necessary and sufficiency conditions. These observations are the following: 1) The quantity to be extremized must contain at least one state parameter. (Otherwise the constraint equations would have no bearing on the problem, i.e., $f$ could then be extremized independently of the equations of constraint.) Thus, at least some multipliers will be defined by Eq. (15). 2) The conditions given under either option 1 or 2 are identical to those required for the function $G(y_i,u_k)$, subject to $g_j$, to be a minimum for the proper choice of multipliers.

***A multiplier rule***

If $y_i^p$, $u_k^p$ is an ordinary extremal point of $f(y_i,u_k)$ subject to $g_j(y_i,u_k)$ both of class $C^2$ and if

$$\partial g_i/\partial y_i|y_i^p,u_k^p \neq 0 \tag{20}$$

then there exists multipliers $\lambda_j$, such that $y_i^p$, $u_k^p$ is also an ordinary extremal point of the function $G(y_i,u_k) = f(y_i,u_k) + \lambda_j g_j(y_i,u_k)$, subject to $g_j(y_i,u_k)$.§

It also follows that an alternate and weaker form of this rule which allows $G$ to be thought of as a function $\lambda_j$ is that there exists multipliers $\lambda_j$ not all zero, such that $y_i^p$, $u_k^p$ is a stationary point of the function $G(y_i,u_k,\lambda_j) = f(y_i,u_k) + \lambda_j g_j(y_i,u_k)$ subject to no constraints.

## III. Conclusions

It is concluded that Lagrange multipliers do not possess the properties sometimes ascribed with their use. Using them to adjoin constraint equations to the quantity to be extremized to form an augmented function does not imply that optimizing conditions for the augmented function with no constraints are equivalent to those of the original problem. The Lagrange multipliers do not in effect unconstrain the constrained variables in the problem. Even though the necessary and sufficient conditions can be expressed completely without the use of Lagrange multipliers, if so desired, their use does allow for a convenient formalism in terms of a multiplier rule.

§ It could be argued that this result is obviously true from the onset, since for any point $y_i$, $u_k$ satisfying $g_j = 0$, the value of $f$ is equal to the value of $G$.

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# Use of Rouse's Stability Parameter in Determining the Critical Layer Height of a Laminar Boundary Layer

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### Nomenclature

| | | |
|---|---|---|
| $f_w$ | = | similarity parameter evaluated at wall conditions |
| $He$ | = | freestream total enthalpy |
| $h_w$ | = | enthalpy based on wall temperature |
| $j$ | = | 1 axisymmetric flow; 0 two-dimensional flow |
| $Me$ | = | Mach number at edge of boundary layer |
| $Pr$ | = | Prandtl number |
| $R$ | = | Rouse's stability parameter |
| $R_{e,x}$ | = | Reynolds number based on boundary-layer edge conditions and distance from leading edge or stagnation point |
| $T_0$ | = | total temperature |
| $t_w$ | = | $h_w/He$ |
| $u$ | = | velocity in $x$ direction |
| $x,y$ | = | coordinate axes |
| $y_c$ | = | critical height |
| $\beta$ | = | pressure gradient parameter |
| $\gamma$ | = | ratio of specific heats |
| $\delta$ | = | boundary-layer thickness |
| $\rho$ | = | density |
| $\mu$ | = | dynamic viscosity |

In connection with boundary-layer transition studies, several investigators[1–7] have measured the location of the maximum mean square output of hot wire or hot film anemometers in laminar boundary layers. The location of this maximum output has been assumed to represent the critical layer in the boundary layer where transition is most likely to be initiated. The characteristics of this critical layer have been used to "explain" some of the properties of boundary-

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